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Correlation Inequalities for Vector Spin Models

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Correlation inequalities for n-vector spin models ($n \geqslant 2$) are reviewed. A relatively simple and unified derivation of the inequalities is achieved, using duplicate variable methods, for spin dimensionalities $n = 2$ (plane rotator model), $n = 3$ (classical Heisenberg model), and $n = 4$. Although correlation inequalities are lacking for $n > 4$, new proofs are presented for the comparison inequalities relating correlations for systems with arbitrary spin dimensionality to corresponding correlations for systems with low spin dimensionality ($n = 1$ or 2).

KEY WORDS: Correlation inequalities; n-vector model.

1. INTRODUCTION

Griffiths[1] first introduced correlation inequalities for spin-$\frac{1}{2}$ Ising ferromagnets in 1967. Today these remarkable inequalities represent an enormously useful and powerful tool in the study of a variety of magnetic lattice spin models. Thus considerable interest centers, first, on finding further inequalities, and second, on extending the inequalities to the largest possible class of models. The search for new inequalities has proved particularly fruitful for the scalar ($n = 1$) Ising systems, and recently Sylvester has written an excellent review[2] on correlation inequalities for general continuous-spin Ising models. Although many inequalities are now known for other systems, no comparable review exists describing the recent advances in this area. Our aim here is to partly fill this gap by reviewing the current status of correlation inequalities for vector spin systems.

Over the past few years the duplicate variable method has emerged as the most important technique for proving correlation inequalities. In this

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paper we discard many of the original proofs and develop proofs based on duplicate variables. We believe an overall unification and simplification is achieved in this area of statistical mechanics by using the duplicate variable method and we feel that this paper, along with Sylvester's, confirms this belief.

In the remainder of this introduction we indicate the layout of the paper. For interest we intersperse some remarks of a historical nature. Since we do not review the many possible applications of the various correlation inequalities, the reader interested in these matters is referred to the original articles.

In Section 2 we introduce our notation and prove two peripheral lemmas. We also prove a simple inequality, expressing the nonnegativity of correlations, which holds for arbitrary ferromagnetic n-vector spin systems. For $n = 1$ this inequality is just the first Griffiths–Kelly–Sherman inequality[1,3] for the Ising model.

Inequalities for systems with two-dimensional (plane rotator) spins are treated in Section 3. The main results are obtained as corollaries to Theorems 3.2 and 3.7. The component-wise correlation inequalities (Corollary 3.4) were obtained for pair interactions by Monroe[4] and extended to many-body interactions by Kunz *et al.*[5] and Dunlop.[6] The Gaussian inequalities (Corollary 3.5) analogous to Newman's inequalities[7] for the Ising model have been proved by Bricmont.[8] The vector-coupling inequalities (Theorem 3.7, Corollaries 3.8 and 3.9) are essentially due to Ginibre.[9] The inequalities of Corollaries 3.3 and 3.8 were obtained by Messager *et al.*[10] and used to obtain results on the uniqueness of the equilibrium state for the plane rotator model (see also Bricmont *et al.*[11]). The special form and application of these inequalities were inspired by the work of Lebowitz[12] on Ising models. The final results (Theorem 3.10 and Corollary 3.11) of Section 3 are new and give some information on the decay of the plane rotator correlation functions. The form of these inequalities was suggested by the Ising model inequalities of Schrader,[13] Messager and Miracle-Sole,[14] and Hegerfeldt.[15]

Section 4 contains inequalities for systems with higher dimensional spins. The main results for $n = 3$ and $n = 4$ (Corollaries 4.2, 4.3, 4.5, and 4.6) are due to Kunz *et al.*[5] and Dunlop.[6] Theorems 4.8 and 4.9 offer a restatement and new proof of the comparison inequalities of Thompson,[16] Bricmont,[17] and Kunz *et al.*[5]

In Section 5 we conclude with a discussion of some open problems.

Finally, for completeness we mention that a number of correlation inequalities are known to hold[8,18] for discrete rotators. Although we have not written out any inequalities for such models explicitly, analogs of many of our results are in fact easily obtained for discrete rotators as consequences of our main theorems. The methods presented here can also be used to obtain inequalities for lattice models in quantum field theory. Although not included in this review, many of these inequalities can be found in the references cited.

2. NOTATION AND PRELIMINARY RESULTS

Let Λ be a finite set of sites. To each site $i \in \Lambda$ associate an n-dimensional vector (classical spin) $\mathbf{S}_i = (S_i^1, S_i^2, \ldots, S_i^n) \in \mathbb{R}^n$ of some fixed length, let us say unit length. These spins are then naturally parametrized by the points of the unit sphere S^n in $\mathbb{R}^n$, and the configuration space for the system is $\{\mathbf{S}\} = \otimes_{i\in\Lambda} S^n$. A duplicate system is formed by associating an additional unit vector (spin) $\bar{\mathbf{S}}_i \in \mathbb{R}^n$ to each site $i \in \Lambda$. The configuration space for the doubled system is then $\{\mathbf{S}, \bar{\mathbf{S}}\} = \otimes_{i\in\Lambda} (S^n \otimes S^n)$.

To discuss the various Hamiltonians it is convenient to introduce the set

$$\mathscr{M} = \mathbb{Z}^\Lambda = \{M: \quad \Lambda \to \mathbb{Z}\} \tag{2.1}$$

of all multiplicity functions on Λ, and the subset

$$\mathscr{M}^+ = \{A \in \mathscr{M}: \quad A(i) \geqslant 0 \text{ for all } i \in \Lambda\} \tag{2.2}$$

of all nonnegative multiplicity functions on Λ. For clarity, we use M, N, etc., in the sequel to denote arbitrary elements of $\mathscr{M}$, and A, B, C, D, etc. to denote elements of $\mathscr{M}^+$. Given $A \in \mathscr{M}^+$ and a variable x taking the values x_i for $i \in \Lambda$, we set

$$x_A = \prod_{i\in\Lambda} x_i^{A(i)} \tag{2.3}$$

In this notation the most general Hamiltonian for a vector spin model is

$$H(\mathbf{S}) = - \sum_{A,B,\ldots G\in\mathscr{M}^+} J_{AB\cdots G} S_A{}^1 S_B{}^2 \cdots S_G{}^n \tag{2.4}$$

This represents the energy of the system in the configuration $\mathbf{S}$. If the interaction parameters $J_{AB\cdots G}$ are such that the energy assignments favor parallel alignment of the spins, then the Hamiltonian (2.4) is called ferromagnetic. The usual pair-interaction Hamiltonian is

$$H(\mathbf{S}) = - \sum_{i,j\in\Lambda} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_{i\in\Lambda} h_i S_i^1 \tag{2.5}$$

where, according to convention, we use h_i for the single-site (magnetic field) parameters. This Hamiltonian is ferromagnetic if all the parameters J_{ij} and h_i are nonnegative. The special cases $n = 1$, 2, and 3 of Eqs. (2.4) and (2.5) refer to the Ising, plane rotator, and classical Heisenberg models, respectively.

The expectation value, i.e., the usual thermal average, of a function $f(\mathbf{S})$ on $\{\mathbf{S}\}$ is defined by

$$\langle f\rangle_n = \int_{\{\mathbf{S}\}} fe^{-H}\, d\mathbf{S} \Big/ \int_{\{\mathbf{S}\}} e^{-H}\, d\mathbf{S} \tag{2.6}$$

where $d\mathbf{S} = \prod_{i\in\Lambda} d\mathbf{S}_i$ indicates an integration over configurations. We extend this definition to the doubled system, described by the Hamiltonian $\mathscr{H}(\mathbf{S}, \bar{\mathbf{S}})$, by setting

$$\langle f\rangle_n = \int_{\{\mathbf{S},\bar{\mathbf{S}}\}} fe^{-\mathscr{H}}\, d\mathbf{S}\, d\bar{\mathbf{S}} \Big/ \int_{\{\mathbf{S},\bar{\mathbf{S}}\}} e^{-\mathscr{H}}\, d\mathbf{S}\, d\bar{\mathbf{S}} \tag{2.7}$$

for any function $f(\mathbf{S}, \bar{\mathbf{S}})$ on $\{\mathbf{S}, \bar{\mathbf{S}}\}$. Clearly, this agrees with the previous definition for functions $f(\mathbf{S})$ when the duplicate systems are independent, that is, when the doubled Hamiltonian is of the form

$$\mathscr{H}(\mathbf{S}, \bar{\mathbf{S}}) = H(\mathbf{S}) + \bar{H}(\bar{\mathbf{S}}) \tag{2.8}$$

Note that, since the inverse temperature $\beta = 1/kT$ plays no significant role, we have set it equal to unity.

Theorem 2.1. Suppose the Hamiltonian for a vector spin system can be written in the form (2.4) with all the parameters $J_{AB\cdots G} \geqslant 0$. Then

$$\langle S_A{}^1 S_B{}^2 \cdots S_G{}^n\rangle_n \geqslant 0 \tag{2.9}$$

for all $A, B,\ldots, G \in \mathscr{M}^+$.

Proof. Since the partition function in (2.6) is nonnegative, we only need to show

$$\int_{\{\mathbf{S}\}} S_A{}^1 S_B{}^2 \cdots S_G{}^n \exp[-H(\mathbf{S})]\, d\mathbf{S} \geqslant 0 \tag{2.10}$$

By expanding the exponential and integrating term by term, our task reduces to proving the inequality

$$\int_{S^n} (S_i{}^1)^p (S_i{}^2)^q \cdots (S_i{}^n)^t\, d\mathbf{S}_i \geqslant 0 \tag{2.11}$$

for all nonnegative integers $p, q,\ldots, t$. But now the integral (2.11) vanishes by symmetry ($S_i{}^1 \to -S_i{}^1$ etc.) unless $p, q,\ldots, t$ are all even, in which case the integral is trivially nonnegative.

The correlation inequality (2.9) is an inequality of the first kind. To obtain inequalities of the second kind we use duplicate variables. To obtain these inequalities for higher dimensional spins we decompose the spins into coupled two-dimensional spins. Because two-dimensional spins enter in this special role we denote them by $\boldsymbol{\sigma}$, $\bar{\boldsymbol{\sigma}}$, $\boldsymbol{\tau}$, $\bar{\boldsymbol{\tau}}$, etc. The subscript 2 in (2.6) and

(2.7) labeling expectations of functions of such variables is then redundant and will henceforth be dropped.

To deal with two-dimensional spins it is convenient to introduce a number of auxiliary variables. Given $i \in \Lambda$ we define variables

$$\begin{aligned} \alpha_i &= \sigma_i^1 + \bar{\sigma}_i^1, & \beta_i &= \sigma_i^1 - \bar{\sigma}_i^1 \\ \gamma_i &= \bar{\sigma}_i^2 + \sigma_i^2, & \delta_i &= \bar{\sigma}_i^2 - \sigma_i^2 \end{aligned} \tag{2.12}$$

It is also useful to introduce angular variables. Using the parametrization

$$\sigma_i^1 = \cos\phi_i, \qquad \sigma_i^2 = \sin\phi_i \tag{2.13}$$

with $0 \leqslant \phi_i < 2\pi$ for $i \in \Lambda$, we find

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \cos(\phi_i - \phi_j) \tag{2.14}$$

This motivates consideration of functions generated by the variables $\cos(M\cdot\phi)$ with $M \in \mathscr{M}$ and

$$M\cdot\phi = \sum_{i\in\Lambda} M(i)\phi_i \tag{2.15}$$

A polynomial in the variables x_i, $i \in \Lambda$, is a sum $\sum_{A\in\mathscr{M}^+} a_A x_A$ of monomials (2.3) with only a finite number of the coefficients $a_A \neq 0$. The set P of all such polynomials with nonnegative coefficients ($a_A \geqslant 0$) is a multiplicative positive cone, i.e., a convex cone closed under multiplication. In particular, af, $f + g$, $fg \in P$ whenever $f, g \in P$ and $a \geqslant 0$. We say that the cone P is generated by the variables x_i, $i \in \Lambda$. Given cones P and Q, we define $P + Q = \{f + g : f \in P, g \in Q\}$ and $PQ = \{\sum_{s=1}^n f_s g_s : f_s \in P, g_s \in Q;\ n = 1, 2, 3,\ldots\}$. For ease of reference we list various cones and their generators in Table I. These cones will be used in the sequel without further specification. Special Roman letters denote cones generated by variables defined on doubled systems.

Table I

Cone	Generators
P_1	σ_i^1; $i \in \Lambda$
P_2	σ_i^2; $i \in \Lambda$
$\mathbb{P}_1$	α_i, β_i; $i \in \Lambda$
$\mathbb{P}_2$	γ_i, δ_i; $i \in \Lambda$
$\mathbb{P} = \mathbb{P}_1\mathbb{P}_2$	$\alpha_i, \beta_i, \gamma_i, \delta_i$; $i \in \Lambda$
Q	$\cos(M\cdot\phi)$; $M \in \mathscr{M}$
$\mathbb{Q}$	$\cos(M\cdot\phi) \pm \cos(M\cdot\bar{\phi})$; $M \in \mathscr{M}$
$\mathbb{Q}_1$	$\cos(M\cdot\phi^1) \pm \cos(M\cdot\bar{\phi}^1)$; $M \in \mathscr{M}$
$\mathbb{Q}_2$	$\cos(M\cdot\bar{\phi}^2) \pm \cos(M\cdot\phi^2)$; $M \in \mathscr{M}$

We conclude this section with two peripheral results that we will need in later sections.

Lemma 2.2. Let x and y be variables defined on Λ. Then if $A \in \mathscr{M}^+$,

$$x_A \pm y_A = 2^{-|A|+1} \sum_{\substack{B+C=A \\ |B| \text{even(odd)}}} (x-y)_B (x+y)_C \frac{A!}{B!\,C!} \tag{2.16}$$

where

$$|A| = \sum_{i\in\Lambda} |A(i)|, \qquad A! = \prod_{i\in\Lambda} A(i)! \tag{2.17}$$

and the sum of B and C in $\mathscr{M}^+$ is given by

$$(B + C)(i) = B(i) + C(i) \tag{2.18}$$

In particular,

$$\begin{aligned} x_i x_j + y_i y_j &= \tfrac{1}{2}[(x_i + y_i)(x_j + y_j) + (x_i - y_i)(x_j - y_j)] \\ &= \tfrac{1}{2}[(y_i + x_i)(y_j + x_j) + (y_i - x_i)(y_j - x_j)] \end{aligned} \tag{2.19}$$

$$x_i x_j - y_i y_j = \tfrac{1}{2}[(x_i + y_i)(x_j - y_j) + (x_i - y_i)(x_j + y_j)] \tag{2.20}$$

Proof. Write $x_i = \frac{1}{2}[(x_i + y_i) + (x_i - y_i)]$, $y_i = \frac{1}{2}[(x_i + y_i) - (x_i - y_i)]$ for each $i \in \Lambda$ and expand the left-hand side of (2.16).

Lemma 2.3. Let

$$H(\boldsymbol{\sigma}) = -\sum_{A\in\mathscr{M}^+} (J_A{}^1 \sigma_A{}^1 + J_A{}^2 \sigma_A{}^2) \tag{2.21}$$

Then, using the parametrization (2.13), $-H \in Q$ if $|J_A{}^2| \leqslant J_A{}^1$ for all $A \in \mathscr{M}^+$ and $J_A{}^2 = 0$ for $|A|$ odd.

Proof. Write

$$J_A{}^1 \sigma_A{}^1 + J_A{}^2 \sigma_A{}^2 = (J_A{}^1 - |J_A{}^2|)\sigma_A{}^1 + |J_A{}^2|(\sigma_A{}^1 \pm \sigma_A{}^2) \tag{2.22}$$

The first term is trivially an element of Q. The second term vanishes unless $|A|$ is even. In this case it can be brought into the required form using Lemma 2.2 and the identity

$$\cos\phi_i \cos\phi_j \pm \sin\phi_i \sin\phi_j = \cos(\phi_i \mp \phi_j) \tag{2.23}$$

3. INEQUALITIES FOR TWO-DIMENSIONAL SPINS

In this section we derive various correlation inequalities for systems with two-dimensional spins. We begin by duplicating the spin variables; this reduces the problem to proving an inequality of lower order for the doubled system. The fundamental results are most easily stated in terms of the auxiliary variables introduced in the previous section.

Lemma 3.1. Let $d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ be any measure on $\{\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}\}$ and consider the $3|\Lambda|$ symmetries (i) $\boldsymbol{\sigma}_i \leftrightarrow \bar{\boldsymbol{\sigma}}_i$, (ii) $\sigma_i^1 \to -\sigma_i^1$, $\bar{\sigma}_i^1 \to -\bar{\sigma}_i^1$, (iii) $\sigma_i^2 \to -\sigma_i^2$, $\bar{\sigma}_i^2 \to -\bar{\sigma}_i^2$; $i \in \Lambda$. Then the inequality

$$\int_{\{\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}\}} \alpha_A \beta_B \gamma_C \delta_D \, d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) \geqslant 0 \tag{3.1}$$

holds for all $A, B, C, D \in \mathscr{M}^+$ in the following three cases:

1. $d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ is invariant under the symmetries (i)–(iii).
2. $d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ is invariant under the symmetries (i) and (ii) and $d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = 0$ unless $\sigma_i^2, \bar{\sigma}_i^2 \geqslant 0$ for all $i \in \Lambda$.
3. $d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ is invariant under the symmetries (i) and $d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = 0$ unless $\sigma_i^1, \bar{\sigma}_i^1, \sigma_i^2, \bar{\sigma}_i^2 \geqslant 0$ for all $i \in \Lambda$.

Proof. We prove the result for case 1. In this case, by the assumed symmetries (i)–(iii), the integral (3.1) vanishes unless the four multiplicities $A(i)$, $B(i)$, $C(i)$, and $D(i)$ all have the same parity at each site $i \in \Lambda$. If the parity is even at a particular site, that site clearly contributes a nonnegative factor to the integrand. On the other hand, if the parity is odd at the site in question, it is reduced to even parity by factoring off the single nonnegative term

$$\alpha_i \beta_i \gamma_i \delta_i = [(\sigma_i^1)^2 - (\bar{\sigma}_i^1)^2]^2 \tag{3.2}$$

So again the site contributes a nonnegative factor to the integrand. This establishes (3.1) for case 1. It is established in the other cases, which we will need in the next section, by similar arguments.

Theorem 3.2. Suppose the Hamiltonian for a duplicated system can be written in the form

$$\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = -\sum_{A,B,C,D \in \mathscr{M}^+} J_{ABCD} \alpha_A \beta_B \gamma_C \delta_D \tag{3.3}$$

with all the coefficients $J_{ABCD} \geqslant 0$, i.e., $-\mathscr{H} \in \mathbb{P}$. Then if $G(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ is a nonnegative function invariant under the $3|\Lambda|$ symmetries of Lemma 3.1, the inequality

$$\langle fG \rangle \geqslant 0 \tag{3.4}$$

holds for all $f \in \mathbb{P}$.

Proof. It clearly suffices to consider the case when f is a monomial. We want to show

$$\int_{\{\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}\}} \alpha_A \beta_B \gamma_C \delta_D \exp[-\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})] \, d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) \geqslant 0 \tag{3.5}$$

where

$$d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = G(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) \prod_{i \in \Lambda} d\boldsymbol{\sigma}_i \, d\bar{\boldsymbol{\sigma}}_i \tag{3.6}$$

But now, from (3.3), the exponential can be expanded as a series of monomials in the α, β, γ, and δ variables with nonnegative coefficients. Thus the result follows by term-by-term integration, using Lemma 3.1, because the measure (3.6) is invariant under the required symmetries.

Corollary 3.3. Let

$$H(\boldsymbol{\sigma}) = -\sum_{A\in\mathscr{M}^+} (J_A{}^1\sigma_A{}^1 + J_A{}^2\sigma_A{}^2) \tag{3.7}$$

$$\bar{H}(\bar{\boldsymbol{\sigma}}) = -\sum_{A\in\mathscr{M}^+} (\bar{J}_A{}^1\bar{\sigma}_A{}^1 + \bar{J}_A{}^2\bar{\sigma}_A{}^2) \tag{3.8}$$

be Hamiltonians for two independent systems defined on Λ. If $J_A{}^1 \geqslant |\bar{J}_A{}^1|$ and $\bar{J}_A{}^2 \geqslant |J_A{}^2|$ for all $A \in \mathscr{M}^+$, then

$$\langle\sigma_A{}^1\rangle - \langle\bar{\sigma}_A{}^1\rangle \geqslant |\langle\sigma_A{}^1\sigma_B{}^1\rangle\langle\bar{\sigma}_B{}^1\rangle - \langle\bar{\sigma}_A{}^1\bar{\sigma}_B{}^1\rangle\langle\sigma_B{}^1\rangle| \tag{3.9}$$

$$\langle\bar{\sigma}_A{}^2\rangle - \langle\sigma_A{}^2\rangle \geqslant |\langle\sigma_A{}^2\sigma_B{}^2\rangle\langle\bar{\sigma}_B{}^2\rangle - \langle\bar{\sigma}_A{}^2\bar{\sigma}_B{}^2\rangle\langle\sigma_B{}^2\rangle| \tag{3.10}$$

for any $A, B \in \mathscr{M}^+$. The expectations can be taken in the appropriate individual system, that is (3.7) or (3.8), or equivalently in the doubled system

$$\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = H(\boldsymbol{\sigma}) + \bar{H}(\bar{\boldsymbol{\sigma}}) \tag{3.11}$$

Proof. The proofs of the inequalities (3.9) and (3.10) are similar. To obtain (3.9), write

$$\begin{aligned}\langle\sigma_A{}^1\rangle - \langle\bar{\sigma}_A{}^1\rangle \pm [\langle\sigma_A{}^1\sigma_B{}^1\rangle\langle\bar{\sigma}_B{}^1\rangle - \langle\bar{\sigma}_A{}^1\bar{\sigma}_B{}^1\rangle\langle\sigma_B{}^1\rangle]&\\ = \langle(\sigma_A{}^1 - \bar{\sigma}_A{}^1)(1 \pm \sigma_B{}^1\bar{\sigma}_B{}^1)\rangle&\end{aligned} \tag{3.12}$$

Now use Lemma 2.2 to show $\sigma_A{}^1 - \bar{\sigma}_A{}^1 \in \mathbb{P}_1$. Taking $G(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = 1 \pm \sigma_B{}^1\bar{\sigma}_B{}^1$, we find that the required result then follows from Theorem 3.2. The hypotheses of this theorem are satisfied because

$$\begin{aligned}-\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = \tfrac{1}{2} \sum_{A\in\mathscr{M}^+} [&(J_A{}^1 + \bar{J}_A{}^1)(\sigma_A{}^1 + \bar{\sigma}_A{}^1) + (J_A{}^1 - \bar{J}_A{}^1)(\sigma_A{}^1 - \bar{\sigma}_A{}^1)\\ &+ (\bar{J}_A{}^2 + J_A{}^2)(\bar{\sigma}_A{}^2 + \sigma_A{}^2) + (\bar{J}_A{}^2 - J_A{}^2)(\bar{\sigma}_A{}^2 - \sigma_A{}^2)]\end{aligned} \tag{3.13}$$

is an element of $\mathbb{P}$ by Lemma 2.2.

Corollary 3.4. Let

$$H(\boldsymbol{\sigma}) = -\sum_{A\in\mathscr{M}^+} (J_A{}^1\sigma_A{}^1 + J_A{}^2\sigma_A{}^2) \tag{3.14}$$

with $J_A{}^1 \geqslant 0$ and $J_A{}^2 \geqslant 0$ for all $A \in \mathscr{M}^+$ so that $-H \in P_1 + P_2$. Then

$$\partial\langle\sigma_A{}^1\rangle/\partial J_B{}^1 \equiv \langle\sigma_A{}^1\sigma_B{}^1\rangle - \langle\sigma_A{}^1\rangle\langle\sigma_B{}^1\rangle \geqslant 0 \tag{3.15}$$

$$\partial\langle\sigma_A{}^1\rangle/\partial J_B{}^2 \equiv \langle\sigma_A{}^1\sigma_B{}^2\rangle - \langle\sigma_A{}^1\rangle\langle\sigma_B{}^2\rangle \leqslant 0 \tag{3.16}$$

Proof. From Corollary 3.3 for a duplicated system,

$$(\langle\sigma_A{}^1\rangle - \langle\bar{\sigma}_A{}^1\rangle)/(J_B{}^1 - \bar{J}_B{}^1) \geqslant 0 \tag{3.17}$$

$$(\langle\sigma_A{}^1\rangle - \langle\bar{\sigma}_A{}^1\rangle)/(J_B{}^2 - \bar{J}_B{}^2) \leqslant 0 \tag{3.18}$$

provided $J_A{}^1 > \bar{J}_A{}^1 \geqslant 0$ and $\bar{J}_A{}^2 > J_A{}^2 \geqslant 0$ for all $A \in \mathscr{M}^+$. Now take the limits $\bar{J}_A{}^1 \to J_A{}^1 -$ and $\bar{J}_A{}^2 \to J_A{}^2 +$ for each $A \in \mathscr{M}^+$. Alternatively, from Theorem 3.2 and Lemma 2.2

$$\langle(\sigma_A{}^1 - \bar{\sigma}_A{}^1)(\sigma_B{}^1 - \bar{\sigma}_B{}^1)\rangle \geqslant 0, \qquad \langle(\sigma_A{}^1 - \bar{\sigma}_A{}^1)(\bar{\sigma}_B{}^2 - \sigma_B{}^2)\rangle \geqslant 0 \tag{3.19}$$

Setting $J_A = \bar{J}_A$ for all $A \in \mathscr{M}^+$, we find that these are precisely the desired inequalities (3.15) and (3.16).

The results of Corollary 3.4 can be used to establish Gaussian inequalities for plane rotators. Since the proof, involving induction, is specialized, we merely state the results. The detailed proof can be found in the paper by Bricmont.[8] Given $A \in \mathscr{M}^+$, we define $p = \{B_s\colon s = 1, 2,..., n\}$ to be a partition of A if

$$A = \sum_{s=1}^{n} B_s \tag{3.20}$$

and each $B_s \in \mathscr{M}^+$. A partition p is called a pair partition if either $|B_s| = 2$ for all s (when $|A|$ is even) or $|B_t| = 1$ for a single t and $|B_s| = 2$ for all $s \neq t$ (when $|A|$ is odd).

Corollary 3.5. Given the Hamiltonian of Corollary 3.4, with the additional symmetry $J_A{}^1 = J_A{}^2$ for all $A \in \mathscr{M}^+$, then

$$\langle\sigma_A{}^1\rangle \leqslant \sum_p \prod_{s=1}^{n} \langle\sigma_{B_s}^1\rangle \tag{3.21}$$

where the sum extends over all pair partitions of A.

So far we have derived componentwise inequalities, that is, we have taken the components of the spins to be the basic variables. We now derive different inequalities in terms of the variables $\cos(M\cdot\phi)$ introduced in Section 2. The connection between these alternative parametrizations is given by (2.13). The fundamental result in terms of duplicate variables is the following.

Lemma 3.6. The inequality

$$\int_0^{2\pi} d\phi\, d\bar{\phi} \prod_{s=1}^{n} [\cos(M_s\cdot\phi) \pm \cos(M_s\cdot\bar{\phi})] \geqslant 0 \tag{3.22}$$

(where $\int_0^{2\pi} d\phi$ stands for $\int_0^{2\pi} \cdots \int \prod_{i\in\Lambda} d\phi_i$) holds for any $M_1, M_2,..., M_n \in \mathscr{M}$ and for any sequence of plus or minus signs.

Proof. For any $M \in \mathscr{M}$,

$$\begin{aligned} \cos(M\cdot\phi) + \cos(M\cdot\bar{\phi}) &= 2\cos(M\cdot\Phi)\cos(M\cdot\bar{\Phi}) \\ \cos(M\cdot\phi) - \cos(M\cdot\bar{\phi}) &= 2\sin(M\cdot\Phi)\sin(M\cdot\bar{\Phi}) \end{aligned} \tag{3.23}$$

where

$$\Phi_i = \tfrac{1}{2}(\phi_i + \bar{\phi}_i), \qquad \bar{\Phi}_i = \tfrac{1}{2}(\bar{\phi}_i - \phi_i) \tag{3.24}$$

Substituting (3.23) into (3.22), we obtain an expression of the form $F(\Phi)F(\bar{\Phi})$ for the integrand. Using periodicity and changing the integration variables,[8] we can then write the integral as

$$\int_{-\pi}^{\pi} d\phi\, d\bar{\phi}\, F(\Phi)F(\bar{\Phi}) = \left[\int_{-\pi}^{\pi} d\Phi\, F(\Phi)\right]^2 \geqslant 0 \tag{3.25}$$

Theorem 3.7. Let $\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})$ be a Hamiltonian for a duplicated system defined on Λ. Then if $-\mathscr{H} \in \mathbb{Q}$,

$$\langle f \rangle \geqslant 0 \tag{3.26}$$

for all $f \in \mathbb{Q}$.

Proof. The result follows by applying Lemma 3.6 to the terms of the series obtained by expanding the Boltzmann factor $\exp[-\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}})]$ and f.

Corollary 3.8. Let

$$H(\boldsymbol{\sigma}) = -\sum_{M\in\mathscr{M}} J_M \cos(M\cdot\phi) \tag{3.27}$$

$$\bar{H}(\bar{\boldsymbol{\sigma}}) = -\sum_{M\in\mathscr{M}} \bar{J}_M \cos(M\cdot\bar{\phi}) \tag{3.28}$$

be Hamiltonians for two independent systems defined on Λ. If $J_M \geqslant |\bar{J}_M|$ for all $M \in \mathscr{M}$, then

$$\begin{aligned} &\langle\cos(M\cdot\phi)\cos(N\cdot\phi)\rangle - \langle\cos(M\cdot\bar{\phi})\cos(N\cdot\bar{\phi})\rangle \\ &\quad \geqslant |\langle\cos(M\cdot\phi)\rangle\langle\cos(N\cdot\bar{\phi})\rangle - \langle\cos(M\cdot\bar{\phi})\rangle\langle\cos(N\cdot\phi)\rangle| \end{aligned} \tag{3.29}$$

for any $M, N \in \mathscr{M}$. The expectations can be taken in the appropriate individual system, that is (3.27) or (3.28), or in the doubled system

$$\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = H(\boldsymbol{\sigma}) + \bar{H}(\bar{\boldsymbol{\sigma}}) \tag{3.30}$$

Proof. We write

$$\begin{aligned} f &= [\cos(M\cdot\phi)\cos(N\cdot\phi) - \cos(M\cdot\bar{\phi})\cos(N\cdot\bar{\phi})] \\ &\quad \pm [\cos(M\cdot\phi)\cos(N\cdot\bar{\phi}) - \cos(M\cdot\bar{\phi})\cos(N\cdot\phi)] \\ &= [\cos(M\cdot\phi) \mp \cos(M\cdot\bar{\phi})][\cos(N\cdot\phi) \pm \cos(N\cdot\bar{\phi})] \end{aligned} \tag{3.31}$$

The result now follows from Theorem 3.7 because

$$-\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = \tfrac{1}{2} \sum_{M \in \mathscr{M}} \{(J_M + \bar{J}_M)[\cos(M\cdot\phi) + \cos(M\cdot\bar{\phi})] + (J_M - \bar{J}_M)[\cos(M\cdot\phi) - \cos(M\cdot\bar{\phi})]\} \tag{3.32}$$

so $-\mathscr{H}$ and f above are both elements of $\mathbb{Q}$.

Corollary 3.9. Let

$$H(\boldsymbol{\sigma}) = -\sum_{M \in \mathscr{M}} J_M \cos(M\cdot\phi) \tag{3.33}$$

be a Hamiltonian with all the parameters $J_M \geqslant 0$, i.e., $-H \in \mathbb{Q}$. Then

$$\frac{\partial}{\partial J_N} \langle\cos(M\cdot\phi)\rangle \equiv \langle\cos(M\cdot\phi)\cos(N\cdot\phi)\rangle - \langle\cos(M\cdot\phi)\rangle\langle\cos(N\cdot\phi)\rangle \geqslant 0 \tag{3.34}$$

for all $M, N \in \mathscr{M}$.

Proof. The result follows in the same manner as Corollary 3.4.

The set of inequalities we derive next uses the idea of duplicate variables in an extended sense. Instead of considering independent duplicate systems, we allow an exchange between the systems. The particular situation we have in mind is the case where the duplicate systems are two halves of a single initial system, symmetric under reflection.

Let Ω be a set of sites in d-dimensional Euclidean space, invariant under some reflection Θ with respect to a fixed $(d-1)$-dimensional hyperplane. Given $i \in \Omega$, let Θi denote the site in Ω obtained from i by this reflection. The set Ω can then be decomposed as

$$\Omega = \Lambda \cup \Theta\Lambda \tag{3.35}$$

where

$$\Theta\Lambda = \{\Theta i: \quad i \in \Lambda\} \tag{3.36}$$

We assume that this decomposition is disjoint and that no sites are left invariant by the reflection. Also we set

$$\sigma_{\Theta i} = \bar{\sigma}_i, \qquad J_{i\Theta j} = \bar{J}_{ij}, \qquad h_{\Theta i} = \bar{h}_i; \qquad i, j \in \Lambda \tag{3.37}$$

Theorem 3.10. Let

$$\mathscr{H}(\boldsymbol{\sigma}) = -\sum_{i,j \in \Omega} (J_{ij}^1 \sigma_i^1 \sigma_j^1 + J_{ij}^2 \sigma_i^2 \sigma_j^2) - \sum_{i \in \Omega} (h_i^1 \sigma_i^1 + h_i^2 \sigma_i^2) \tag{3.38}$$

be the Hamiltonian for a system on the reflection-invariant set Ω. If, in the above notation, the interactions satisfy

$$\text{(i)} \quad J_{ij}^1 = J_{\Theta i \Theta j}^1, \quad J_{ij}^2 = J_{\Theta i \Theta j}^2 \tag{3.39}$$

$$\text{(ii)} \quad J_{ij}^1 \geqslant |\bar{J}_{ij}^1|, \quad \bar{J}_{ij}^2 \geqslant |J_{ij}^2| \tag{3.40}$$

$$\text{(iii)} \quad h_i^1 \geqslant |\bar{h}_i^1|, \quad \bar{h}_i^2 \geqslant |h_i^2| \tag{3.41}$$

then

$$\langle f \rangle \geqslant 0 \tag{3.42}$$

for any $f \in \mathbb{P}$.

Proof. By Theorem 3.2 we need only show that $-\mathscr{H} \in \mathbb{P}$. But now the contribution from the first components, viz.

$$\sum_{i,j\in\Lambda} [J_{ij}^1(\sigma_i^1\sigma_j^1 + \bar{\sigma}_i^1\bar{\sigma}_j^1) + \bar{J}_{ij}^1(\sigma_i^1\bar{\sigma}_j^1 + \bar{\sigma}_i^1\sigma_j^1) + \sum_{i\in\Lambda}(h_i^1\sigma_i^1 + \bar{h}_i^1\bar{\sigma}_i^1)] \tag{3.43}$$

$$= \tfrac{1}{2} \sum_{i,j\in\Lambda} [(J_{ij}^1 + \bar{J}_{ij}^1)\alpha_i\alpha_j + (J_{ij}^1 - \bar{J}_{ij}^1)\beta_i\beta_j]$$

$$+ \tfrac{1}{2} \sum_{i\in\Lambda} [(h_i^1 + \bar{h}_i^1)\alpha_i + (h_i^1 - \bar{h}_i^1)\beta_i] \tag{3.44}$$

is an element of $\mathbb{P}_1$. Similarly, the contribution from the second components is an element of $\mathbb{P}_2$. Hence $-\mathscr{H} \in \mathbb{P}$.

Corollary 3.11. Let $\mathscr{H}(\sigma)$ be as in Theorem 3.10. Then the inequalities

$$\langle \sigma_A^1\sigma_B^1 \rangle - \langle \sigma_A^1\bar{\sigma}_B^1 \rangle \geqslant 0 \tag{3.45}$$

$$\langle \sigma_A^1\sigma_B^2 \rangle - \langle \sigma_A^1\bar{\sigma}_B^2 \rangle \leqslant 0 \tag{3.46}$$

hold for all $A, B \in \mathscr{M}^+$.

Proof. The required inequalities are, respectively,

$$\langle (\sigma_A^1 - \bar{\sigma}_A^1)(\sigma_B^1 - \bar{\sigma}_B^1) \rangle \geqslant 0 \tag{3.47}$$

$$\langle (\sigma_A^1 - \bar{\sigma}_A^1)(\bar{\sigma}_B^2 - \sigma_B^2) \rangle \geqslant 0 \tag{3.48}$$

These follow from Lemma 2.2 and Theorem 3.10.

4. INEQUALITIES FOR HIGHER-DIMENSIONAL SPINS

The known inequalities for higher-dimensional spins are derived by decomposing the spins into coupled two-dimensional spins. This approach is unfortunately limited to spin dimensionalities $n \leqslant 4$. Let us first consider classical Heisenberg spins, that is, three-dimensional spins. In spherical polars these have the familiar parametrization

$$\mathbf{S}_i = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i) \tag{4.1}$$

$$\bar{\mathbf{S}}_i = (\sin\bar{\theta}_i \cos\bar{\phi}_i; \sin\bar{\theta}_i \sin\bar{\phi}_i, \cos\bar{\theta}_i) \tag{4.2}$$

where $0 \leqslant \theta_i, \bar{\theta}_i < \pi$ and $0 \leqslant \phi_i, \bar{\phi}_i < 2\pi$ for $i \in \Lambda$. Thus each spin $\mathbf{S}_i$ is represented by two plane rotators associated with the angles θ_i and ϕ_i. In the sequel we will use the alternative parametrization

$$\mathbf{S}_i = (\sigma_i^1 \cos\phi_i, \sigma_i^1 \sin\phi_i, \sigma_i^2) \tag{4.3}$$

$$\bar{\mathbf{S}}_i = (\bar{\sigma}_i^1 \cos\bar{\phi}_i, \bar{\sigma}_i^1 \sin\bar{\phi}_i, \bar{\sigma}_i^2) \tag{4.4}$$

with

$$\boldsymbol{\sigma}_i = (\sin\theta_i, \cos\theta_i), \qquad \bar{\boldsymbol{\sigma}}_i = (\sin\bar{\theta}_i, \cos\bar{\theta}_i) \tag{4.5}$$

and the auxiliary variables α_i, β_i, γ_i, and δ_i defined as in (2.12).

Theorem 4.1. Let $\mathscr{H}(\mathrm{S}, \bar{\mathrm{S}})$ be a Hamiltonian for a doubled Heisenberg system and suppose that $-\mathscr{H} \in \mathbb{PQ}$ when the spins are parametrized according to (4.3) and (4.4). Then

$$\langle f \rangle_3 \geqslant 0 \tag{4.6}$$

for any $f \in \mathbb{PQ}$.

Proof. Since

$$\int_{S^3} d\mathbf{S}_i = \int_0^{\pi} d\theta_i \int_0^{2\pi} d\phi_i \sin\theta_i$$

the usual expansion techniques reduce our task to proving

$$\int_0^{\pi} d\theta\, d\bar{\theta} \int_0^{2\pi} d\phi\, d\bar{\phi}\; \sigma_\Lambda{}^1 \bar{\sigma}_\Lambda{}^1 \alpha_A \beta_B \gamma_C \delta_D$$

$$\times \prod_{s=1}^{n} [\cos(M_s \cdot \phi) \pm \cos(M_s \cdot \bar{\phi})] \geqslant 0 \tag{4.7}$$

This is just a combination of Lemmas 3.6 and 3.1 with

$$d\mu(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = G(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) \prod_{i \in \Lambda} d\boldsymbol{\sigma}_i\, d\bar{\boldsymbol{\sigma}}_i \tag{4.8}$$

and

$$G(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = \begin{cases} \sigma_\Lambda{}^1 \bar{\sigma}_\Lambda{}^1 & \text{for } \sigma_i^1, \bar{\sigma}_i^1 \geqslant 0 \quad \text{for all } i \in \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

Corollary 4.2. Let $H(\mathrm{S})$ be a Hamiltonian for a Heisenberg system and suppose that $-H \in P_1 Q + P_2$ when the spins are parametrized according to (4.3). Then

$$\langle f_1 g_1 \rangle_3 - \langle f_1 \rangle_3 \langle g_1 \rangle_3 \geqslant 0 \tag{4.10}$$

$$\langle f_2 g_2 \rangle_3 - \langle f_2 \rangle_3 \langle g_2 \rangle_3 \geqslant 0 \tag{4.11}$$

$$\langle f_1 g_2 \rangle_3 - \langle f_1 \rangle_3 \langle g_2 \rangle_3 \leqslant 0 \tag{4.12}$$

for any $f_1, g_1 \in P_1 Q$ and any $f_2, g_2 \in P_2$.

Proof. Duplicate the system and set

$$\mathscr{H}(\mathrm{S}, \bar{\mathrm{S}}) = H(\mathrm{S}) + \bar{H}(\bar{\mathrm{S}}) \tag{4.13}$$

Now notice that, by Lemma 2.2, $\sigma_A{}^1 \cos(M\cdot\phi) \pm \bar{\sigma}_A{}^1 \cos(M\cdot\bar{\phi}) \in \mathbb{P}_1\mathbb{Q}$ and $\bar{\sigma}_A{}^2 \pm \sigma_A{}^2 \in \mathbb{P}_2$ so that $-\mathscr{H} \in \mathbb{P}\mathbb{Q}$. Thus the required results follow from Theorem 4.1 with the appropriate choice of $f \in \mathbb{P}\mathbb{Q}$. For example, (4.12) is obtained by choosing

$$f(\mathrm{S}, \bar{\mathrm{S}}) = [f_1(\mathrm{S}) - f_1(\bar{\mathrm{S}})][g_2(\bar{\mathrm{S}}) - g_2(\mathrm{S})] \tag{4.14}$$

Corollary 4.3. Let

$$H(\mathrm{S}) = -\sum_{A \in \mathscr{M}} (J_A{}^1 S_A{}^1 + J_A{}^2 S_A{}^2 + J_A{}^3 S_A{}^3) \tag{4.15}$$

be a Heisenberg Hamiltonian with $J_A{}^3 \geqslant 0$, $|J_A{}^2| \leqslant J_A{}^1$ for all $A \in \mathscr{M}^+$ and $J_A{}^2 = 0$ for $|A|$ odd. Then

$$\partial\langle S_A{}^\alpha \rangle_3 / \partial J_B{}^\alpha \equiv \langle S_A{}^\alpha S_B{}^\alpha \rangle_3 - \langle S_A{}^\alpha \rangle_3 \langle S_B{}^\alpha \rangle_3 \geqslant 0; \qquad \alpha = 1, 3 \tag{4.16}$$

$$\partial\langle S_A{}^3 \rangle_3 / \partial J_B{}^1 \equiv \langle S_A{}^3 S_B{}^1 \rangle_3 - \langle S_A{}^3 \rangle_3 \langle S_B{}^1 \rangle_3 \leqslant 0 \tag{4.17}$$

for any $A, B \in \mathscr{M}^+$.

Proof. By Lemma 2.3, $-H \in P_1 Q + P_2$. Thus the results are special cases of Corollary 4.2.

Inequalities similar to the above can also be derived for four-dimensional spins. These spins are parametrized by

$$\mathrm{S}_i = (\sigma_i{}^1 \cos\phi_i{}^1, \sigma_i{}^1 \sin\phi_i{}^1, \sigma_i{}^2 \cos\phi_i{}^2, \sigma_i{}^2 \sin\phi_i{}^2) \tag{4.18}$$

$$\bar{\mathrm{S}}_i = (\bar{\sigma}_i{}^1 \cos\bar{\phi}_i{}^1, \bar{\sigma}_i{}^1 \sin\bar{\phi}_i{}^1, \bar{\sigma}_i{}^2 \cos\bar{\phi}_i{}^2, \bar{\sigma}_i{}^2 \sin\bar{\phi}_i{}^2) \tag{4.19}$$

where

$$\boldsymbol{\sigma}_i = (\sin\theta_i, \cos\theta_i), \qquad \bar{\boldsymbol{\sigma}}_i = (\sin\bar{\theta}_i, \cos\bar{\theta}_i) \tag{4.20}$$

and $0 \leqslant \theta_i, \bar{\theta}_i < \pi/2$ and $0 \leqslant \phi_i{}^1, \phi_i{}^2, \bar{\phi}_i{}^1, \bar{\phi}_i{}^2 < 2\pi$; for all $i \in \Lambda$.

Theorem 4.4. Let $\mathscr{H}(\mathrm{S}, \bar{\mathrm{S}})$ be a Hamiltonian for a doubled system with four-dimensional spins and suppose that $-\mathscr{H} \in \mathbb{P}\mathbb{Q}_1\mathbb{Q}_2$ when the spins are parametrized according to (4.18) and (4.19). Then

$$\langle f \rangle_4 \geqslant 0 \tag{4.21}$$

for any $f \in \mathbb{P}\mathbb{Q}_1\mathbb{Q}_2$.

Proof. Notice that

$$\int_{S^4} d\mathrm{S}_i = \int_0^{\pi/2} d\theta_i \int_0^{2\pi} d\phi_i^1 \int_0^{2\pi} d\phi_i^2 \sin\theta_i \cos\theta_i \tag{4.22}$$

The proof now proceeds as for Theorem 4.1, if we take

$$G(\sigma, \bar{\sigma}) = \begin{cases} \sigma_\Lambda^1 \sigma_\Lambda^2 \bar{\sigma}_\Lambda^1 \bar{\sigma}_\Lambda^2 & \text{for} \quad \sigma_i^1, \sigma_i^2, \bar{\sigma}_i^1, \bar{\sigma}_i^2 \geqslant 0 \quad \text{for all } i \in \Lambda \\ 0, & \text{otherwise} \end{cases} \tag{4.23}$$

Corollary 4.5. Let $H(\mathrm{S})$ be a Hamiltonian for a system with four-dimensional spins and suppose that $-H \in P_1Q_1 + P_2Q_2$ when the spins are parametrized according to (4.18). Then

$$\langle f_1 g_1 \rangle_4 - \langle f_1 \rangle_4 \langle g_1 \rangle_4 \geqslant 0 \tag{4.24}$$

$$\langle f_2 g_2 \rangle_4 - \langle f_2 \rangle_4 \langle g_2 \rangle_4 \geqslant 0 \tag{4.25}$$

$$\langle f_1 g_2 \rangle_4 - \langle f_1 \rangle_4 \langle g_2 \rangle_4 \leqslant 0 \tag{4.26}$$

for any $f_1, g_1 \in P_1Q_1$ and any $f_2, g_2 \in P_2Q_2$.

Proof. The results follow from Theorem 4.4 and the arguments of Corollary 4.2.

Corollary 4.6. Let

$$H(\mathrm{S}) = -\sum_{A \in \mathscr{M}^+} (J_A^1 S_A^1 + J_A^2 S_A^2 + J_A^3 S_A^3 + J_A^4 S_A^4) \tag{4.27}$$

be a Hamiltonian for a system with four-dimensional spins such that $|J_A^2| \leqslant J_A^1$, $|J_A^4| \leqslant J_A^3$ for all $A \in \mathscr{M}^+$ and $J_A^2 = J_A^4 = 0$ for $|A|$ odd. Then

$$\partial\langle S_S^\alpha \rangle_4 / \partial J_B^\alpha \equiv \langle S_A^\alpha S_B^\alpha \rangle_4 - \langle S_A^\alpha \rangle_4 \langle S_B^\alpha \rangle_4 \geqslant 0; \qquad \alpha = 1, 3 \tag{4.28}$$

$$\partial\langle S_A^3 \rangle_4 / \partial J_B^1 \equiv \langle S_A^3 S_B^1 \rangle_4 - \langle S_A^3 \rangle_4 \langle S_B^1 \rangle_4 \leqslant 0 \tag{4.29}$$

for any $A, B \in \mathscr{M}^+$.

Proof. By Lemma 2.3, $-\mathscr{H} \in P_1Q_1 + P_2Q_2$. Thus the results are special cases of Corollary 4.5.

Gaussian inequalities of the form (3.21) can also be obtained, for the spin components labeled 1 and 3, for spin dimensionalities $n = 3$ and $n = 4$. Given $J_A^1 = J_A^3$ for all $A \in \mathscr{M}^+$, these inequalities follow by an inductive proof using the results of Corollaries 4.3 and 4.6, respectively. Again the reader is referred to Bricmont[8] for details.

Although the correlation inequalities of this section are expected to hold for $n > 4$, they have not been extended to these systems. However, there are certain comparison inequalities which relate correlations for high-spin-dimensionality systems to corresponding correlations for low-spin-dimensionality systems. We now give a derivation of these inequalities based on mixed duplicate variables.

Lemma 4.7. Let $\mu_i = \pm 1$, $i \in \Lambda$, be a set of Ising spins. Then

$$\sum_{\{\mu\}} \int_{\{S\}} d\mathbf{S} \prod_i (\mu_i \pm S_i^1) \prod_{(j,k)} (\mu_j \mu_k \pm \mathbf{S}_j \cdot \mathbf{S}_k) \geqslant 0 \tag{4.30}$$

for all products over sites $i \in \Lambda$ and pairs $(j, k) \in \Lambda \times \Lambda$ and for any sequence of plus or minus signs.

Proof. The integrals in (4.30) are invariant under the replacement $\mathbf{S}_i \to \mu_i \mathbf{S}_i$ because the absolute value of the Jacobians is unity for these transformations. Thus the result is obtained by noting that: $1 \pm S_i^1 \geqslant 0$; $1 \pm \mathbf{S}_j \cdot \mathbf{S}_k \geqslant 0$; and $\sum_{\{\mu\}} \mu_A \geqslant 0$ for all $A \in \mathscr{M}^+$.

Theorem 4.8. Let

$$H(\mathbf{S}) = -\sum_{i,j \in \Lambda} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_{i \in \Lambda} h_i S_i^1 \tag{4.31}$$

be an n-vector spin Hamiltonian with $J_{ij} \geqslant 0$ and $h_i \geqslant 0$, and for $n = 1$ write

$$H(\mu) = -\sum_{i,j \in \Lambda} J_{ij} \mu_i \mu_j - \sum_{i \in \Lambda} h_i \mu_i \tag{4.32}$$

Then for $n \geqslant 2$,

$$\langle S_A^1 \rangle_n \leqslant \langle \mu_A \rangle_1 \tag{4.33}$$

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle_n \leqslant \langle \mu_i \mu_j \rangle_1 \tag{4.34}$$

for any $A \in \mathscr{M}^+$ and any pair (i, j) of sites.

Proof. Introduce a mixed duplicate Hamiltonian

$$\mathscr{H}(\mu, \mathbf{S}) = H(\mu) + H(S) \tag{4.35}$$

Then in the doubled system, (4.33) and (4.34) are consequences of the more general inequality

$$\left\langle \prod_A (\mu_A - S_A^1) \prod_{(i,j)} (\mu_i \mu_j - \mathbf{S}_i \cdot \mathbf{S}_j) \right\rangle_{1,n} \geqslant 0 \tag{4.36}$$

which follows by expanding the Boltzmann factor and using Lemma 4.7.

Theorem 4.9. Let

$$H(\mathbf{S}) = -\sum_{i,j\in\Lambda} \sum_{\alpha=1}^{n} J_{ij}^{\alpha} S_i^{\alpha} S_j^{\alpha} - \sum_{i\in\Lambda} h_i S_i^1 \tag{4.37}$$

with $J_{ij}^1 \geqslant |J_{ij}^2|$, $J_{ij}^{\alpha} \geqslant 0$ for all $\alpha \neq 2$ and $h_i \geqslant 0$. Then for $n \geqslant 3$,

$$\langle S_A{}^1 \rangle_n \leqslant \langle S_A{}^1 \rangle_2 \tag{4.38}$$

Proof. Introduce a mixed duplicate Hamiltonian

$$\mathscr{H}(\bar{\boldsymbol{\sigma}}, \mathbf{S}) = H(\bar{\boldsymbol{\sigma}}) + H(\mathbf{S}) \tag{4.39}$$

where parametrically

$$\bar{\boldsymbol{\sigma}}_i = (\cos \bar{\phi}_i, \sin \bar{\phi}_i), \qquad \mathbf{S}_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i \mathbf{U}_i) \tag{4.40}$$

with $0 \leqslant \theta_i < \pi/2$; $0 \leqslant \phi_i$, $\bar{\phi}_i < 2\pi$; and $\mathbf{U}_i$ an $(n-2)$-dimensional unit vector spin. In the doubled system we want to prove that

$$\langle \bar{\sigma}_A{}^1 - S_A{}^1 \rangle_{2,n} \geqslant 0 \tag{4.41}$$

But now

$$\begin{aligned} \bar{\sigma}_i{}^1 \pm S_i{}^1 &= \cos \bar{\phi}_i \pm \sin \theta_i \cos \phi_i \\ &= \tfrac{1}{2}[(1 + \sin \theta_i)(\cos \bar{\phi}_i \pm \cos \phi_i) + (1 - \sin \theta_i)(\cos \bar{\phi}_i \mp \cos \phi_i)] \end{aligned} \tag{4.42}$$

Clearly, by Lemmas 2.2 and 2.3, $\bar{\sigma}_A{}^1 \pm S_A{}^1$ and $-\mathscr{H}(\bar{\boldsymbol{\sigma}}, \mathbf{S})$ can be expanded as multinomials with positive coefficients in the variables: $(1 \pm \sin \theta_i)$, $[\cos(M\cdot\bar{\phi}) \pm \cos(M\cdot\phi)]$, and $\cos \theta_i \cos \theta_j \mathbf{U}_i \cdot \mathbf{U}_j$. Thus, noting that

$$\int_{S^n} d\mathbf{S}_i \int_{S^2} d\bar{\boldsymbol{\sigma}}_i = \int_0^{\pi/2} d\theta_i \sin \theta_i (\cos \theta_i)^{n-3} \int_0^{2\pi} d\phi_i \int_0^{2\pi} d\bar{\phi}_i \int_{S^{n-2}} d\mathbf{U}_i \tag{4.43}$$

we obtain the required result by expanding $\exp[-\mathscr{H}(\bar{\boldsymbol{\sigma}}, \mathbf{S})]$ and using Lemma 3.6 and Theorem 2.1, the integrals over the θ variables being trivially non-negative.

5. DISCUSSION

In retrospect, certain marked trends are evident in the inequalities as the spin dimensionality varies. Most importantly, as n increases, the known results become weaker. Perhaps this should not be surprising. However, it does leave a number of questions still to be answered. To give the reader a clear picture of the situation, we conclude by discussing some of the remaining open problems in the field of correlation inequalities. For definiteness, we focus attention on the pair-interaction Hamiltonian (2.5) in the sequel without further comment.

Perhaps the most outstanding problem is to decide whether the following inequality holds:

$$\partial\langle S_i^1\rangle_n/\partial J_{jl} = \langle S_i^1(\mathbf{S}_j\cdot\mathbf{S}_l)\rangle_n - \langle S_i^1\rangle_n\langle \mathbf{S}_j\cdot\mathbf{S}_l\rangle_n \geqslant 0; \qquad n \geqslant 3 \tag{5.1}$$

The monotonicity property (5.1) would have many important consequences. For example, the results of Frohlich *et al.*[19] on the existence of phase transitions for these systems could be extended beyond nearest-neighbor ferromagnetic interactions.

Another inequality whose validity has not been ascertained is the following:

$$\langle S_A^1 S_B^1\rangle_n - \langle S_A^1\rangle_n\langle S_B^1\rangle_n \geqslant 0; \qquad n \geqslant 5 \tag{5.2}$$

This is an obvious extension of the correlation inequalities (3.15), (4.16), and (4.28) and would be expected to have similar applications.

On the basis of the comparison inequalities (4.33) and (4.38) it is reasonable to make the following conjecture:

$$\langle S_A^1\rangle_n \geqslant \langle S_A^1\rangle_{n+1}; \qquad n \geqslant 3 \tag{5.3}$$

Intuitively, one indeed expects the correlations between spins to decrease as the phase space available to the spins becomes larger.

Our final unsolved problem is the following higher order inequality:

$$\begin{aligned}\partial^2\langle S_i^1\rangle_n/\partial h_j\,\partial h_k = {} & \langle S_i^1 S_j^1 S_k^1\rangle_n - \langle S_i^1\rangle_n\langle S_j^1 S_k^1\rangle_n - \langle S_j^1\rangle_n\langle S_i^1 S_k^1\rangle_n \\ & - \langle S_k^1\rangle_n\langle S_i^1 S_j^1\rangle_n + 2\langle S_i^1\rangle_n\langle S_j^1\rangle_n\langle S_k^1\rangle_n \leqslant 0; \\ & n \geqslant 2\end{aligned} \tag{5.4}$$

For $n = 1$ this is the Griffiths–Hurst–Sherman inequality.[20] In a uniform magnetic field $h_i = h$, it implies the concavity of the magnetization as a function of the field h.

The above inequalities are potentially very useful and because no counterexamples exist we can optimistically hope that they are generally valid. On the other hand, it would also be of interest to know if they are incorrect, as this would perhaps indicate that the way we think about these models needs to be revised.

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